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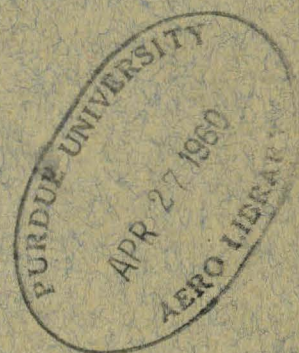
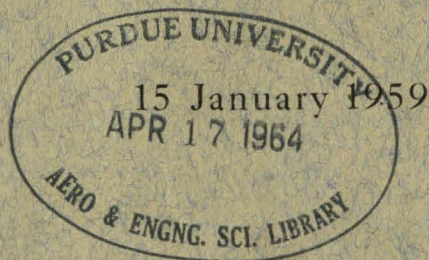
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# FUNDAMENTAL STUDIES RELATING TO SYSTEMS ANALYSIS OF SOLID PROPELLANTS

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Pasadena, California



PROGRESS REPORT NO. 1 - GALCIT 101

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Fundamental Studies Relating to  
Systems Analysis of Solid Propellants

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## I. INTRODUCTION TO THE STRAIN ANALYSIS OF SOLID PROPELLANT GRAINS

In this report the groundwork is laid for the proposed work scope which stressed the need for a greater understanding of the solid mechanics of grains. Particular emphasis will be directed toward the multi-axial behavior of thick walled configurations. The work falls naturally into three areas; (1) analysis procedures, (2) material properties, and (3) failure criteria.

As a necessary preliminary to treating specific designs, certain material of general applicability must be developed, collected, and summarized. The following sections therefore deal with a general description of viscoelastic analysis and material representation, discussed by contrast with more conventional engineering analysis. By this means a background is established for the collection of elastic design formulas which are included in the second section of the report.

### Operational Concept for Stress-Strain Relations

When a steel tensile specimen at a moderate temperature is stretched, it is observed that the instantaneous strain is very nearly proportional to the applied stress up to the yield point; and that if the bar is unloaded gradually the stress follows the same law. Consequently, we say the material obeys Hooke's law in tension up to the yield point and write

$$\sigma = E\epsilon \quad (1)$$

in which  $\sigma$  and  $\epsilon$  are based on the original specimen dimensions as in the usual engineering sense. The proportionality constant  $E$ , which represents the slope of the stress-strain curve, is called Young's modulus or the tensile modulus. Hooke's law can also be written in an inverse form

$$D \sigma = \epsilon \quad (2)$$

The proportionality constant  $D$  is then commonly referred to as the tensile compliance. It has been found experimentally that the above law holds for many materials, particularly metals, as long as the strains are small.

It is evident that if a material which obeys Hooke's law is held at a constant strain, the stress also remains unchanged with time. However when a propellant tensile specimen at room temperature is stretched and held at a constant strain  $\epsilon_0$  (stress relaxation test) the stress  $\sigma(t)$  necessary to maintain this elongation decays with time. In other words, the tensile relaxation modulus  $E(t)$  decreases since it is defined as

$$E(t) = \frac{\sigma(t)}{\epsilon_0} \quad (3)$$

Similarly, if a constant tensile stress  $\sigma_0$  is applied (creep test) the strain increases with time; or equivalently the creep compliance  $D(t)$  increases since

$$D(t) = \frac{\epsilon(t)}{\sigma_0} \quad (4)$$

These two cases are illustrated in Figure 1.

In addition to strong time dependence, the mechanical properties are greatly affected by temperature. Below a temperature  $T_g$ , defined as the glass transition temperature, the propellant is glassy and behaves as a brittle body obeying Hooke's law. Above this temperature the properties are time dependent and vary considerably with temperature.

This behavior leads one to formulate a general functional relationship between tensile stress and strain which includes both time and temperature dependence for temperatures greater than  $T_g$ . However it has been found for many polymers, particularly plastics and rubbers, that these two variables can be considered separately if the temperature range is not too great. For example, if certain material constants are known at one temperature, it is possible to predict behavior at another temperature by simply altering these constants along with shifting the time scale<sup>(1)</sup>. Since composite propellants are filled rubbers or filled plastics and double-base propellants are plastics, it is expected that the same rule should hold. On this basis the time dependent behavior at a fixed temperature will be discussed first, and will be followed by an



explanation of the method used to correct for temperature in a later report.

The relation between stress and strain of a propellant in a tension test at constant temperature can be expressed generally as

$$O_1 \sigma(t) = O_2 \epsilon(t) \quad (5)$$

where  $O_1$  and  $O_2$  represent algebraic and differential operations on  $\sigma(t)$  and  $\epsilon(t)$ . For example, when Hooke's law applies the operators are simple constants

$$O_1 = 1, \quad O_2 = E \quad (6)$$

It is important to note that these operators are not always linear. Indeed, for large deformations of some metals, a more realistic relation is for example<sup>(2)</sup>

$$\sigma = O_2 \epsilon = E(1 - 2\nu\epsilon) \ln(1 + \epsilon) \quad (7)$$

where  $\nu$  is Poisson's ratio. In this instance  $O_2$  is a non-linear algebraic operator.

A simple extension of Hooke's law is to consider stress proportional to both the strain and the strain rate. For this case, equation (5) becomes

$$\sigma(t) = \eta_v \frac{d\epsilon(t)}{dt} + E_v \epsilon(t) \quad (8)$$

in which  $E_v$  and  $\eta_v$  are proportionality constants. An important property of the differential operator is that it is linear and therefore obeys many of the ordinary rules of algebra, such as association, commutation, and superposition. This allows us to write

$$\sigma(t) = \left[ \eta_v \frac{d}{dt} + E_v \right] \epsilon(t) \quad (9)$$

in which

$$O_1 = 1 \quad O_2 = \eta_v \frac{d}{dt} + E_v \quad (10)$$

If we consider a creep test in which a constant stress  $\sigma_0$  is applied to a material that obeys the stress-strain law shown in equation (8), the resulting strain can be calculated by integration. Assuming the specimen to be unstressed and unstrained at time  $t = 0$ , we obtain

$$\epsilon(t) = \frac{\sigma_0}{E_v} \left( 1 - e^{-\frac{t}{\tau_v}} \right) \quad (11)$$

where a retardation time,  $\tau_v$ ,

$$\tau_v = \frac{\eta_v}{E_v} \quad (12)$$

has been defined for convenience.

The integrated expression for  $\epsilon(t)$  in (11) can now be easily checked by substituting it back into equation (9). The creep compliance,  $D(t)$ , as defined by equation (4), is

$$D(t) = \frac{\epsilon(t)}{\sigma_0} = \frac{1}{E_v} \left( 1 - e^{-\frac{t}{\tau_v}} \right) \quad (13)$$

A plot of  $D(t)$  is shown in Figure 2. It is seen that  $D(t)$ , given by equation (13) is similar to the creep compliance shown in Figure 1 for an actual propellant.

However, it is not possible to describe accurately the complete stress and displacement behavior of propellant by such a simple relation; it is therefore necessary to go to more complicated operators. If the degree of complication is such that the stress-strain response can be adequately described by linear operators such as more general forms of (9), then it develops that the mathematics involved in solving stress problems is greatly simplified. This important fact therefore provides the impetus for investigating possible representations of the mechanical behavior by linear operators.

#### Linear Viscoelastic Representation

We now define a special case of equation (5) such that  $O_1$  and  $O_2$  are taken to be linear differential operators with constant coefficients.



In the literature such a representation is called a linear viscoelastic model, and for a simple tensile test is written

$$\left[ p_n \frac{d^n}{dt^n} + \dots p_2 \frac{d^2}{dt^2} + p_1 \frac{d}{dt} + p_0 \right] \sigma(t) = \left[ q_m \frac{d^m}{dt^m} + \dots q_2 \frac{d^2}{dt^2} + q_1 \frac{d}{dt} + q_0 \right] \epsilon(t) \quad (14)$$

or more compactly

$$P'' \sigma(t) = Q'' \epsilon(t) \quad (15)$$

where

$$P'' = \sum_{i=1}^n p_i \frac{d^i}{dt^i} + p_0 \quad (16)$$

$$Q'' = \sum_{i=1}^m q_i \frac{d^i}{dt^i} + q_0 \quad (17)$$

$\frac{d^i}{dt^i}$  is a linear operator that represents the  $i$ th derivative with respect to time and  $p_i$  and  $q_i$  are material constants to be obtained experimentally. The methods used to obtain these constants will be described in a later report.

The above relation (14) has good experimental verification for small strains over a wide temperature range for many unfilled polymers. Now even though composite propellants are essentially highly filled polymers, it is nevertheless expected that they would exhibit more or less of a linear viscoelastic behavior depending upon the specific composition.

The same form of a stress-strain law is found to hold for hydrostatic pressure and volume change, and for shear stress and shear strain in a simple shear test. Thus, the response of an element subjected to hydrostatic pressurization is given by

$$P' \sigma_p(t) = Q' \frac{\Delta v}{v}(t) \quad (18)$$

where  $(\frac{\Delta v}{v}(t))$  is the volume change per unit volume due to the hydrostatic pressure  $\sigma_p(t)$ . Similarly,

$$P \zeta(t) = Q \gamma(t) \quad (19)$$

in which  $\mathcal{T}(t)$  is a shear stress and  $\mathcal{Y}(t)$  the corresponding shear strain.  $P, Q, P', Q'$  are of the form of  $P''$  and  $Q''$  shown in (16) and (17). Equations (15), (18) and (19) are analogous to the elastic stress-strain laws, since for an elastic body undergoing small deformations we can write

$$\begin{aligned} \text{simple tension:} \quad & \sigma = E \epsilon \quad (E = \text{Young's modulus}) \\ \text{hydrostatic pressurization:} \quad & \sigma_p = K \frac{\Delta v}{v} \quad (K = \text{bulk modulus}) \\ \text{shear:} \quad & \mathcal{T} = G \mathcal{Y} \quad (G = \text{shear modulus}) \end{aligned} \quad (20)$$

Similarly for a linear viscoelastic material we have

$$\begin{aligned} \text{simple tension} \quad & \sigma(t) = \frac{Q''}{P''} \epsilon(t) \\ \text{hydrostatic pressurization:} \quad & \sigma_p(t) = \frac{Q'}{P'} \frac{\Delta v(t)}{v} \\ \text{shear:} \quad & \mathcal{T}(t) = \frac{Q}{P} \mathcal{Y}(t) \end{aligned} \quad (21)$$

When it comes to actually computing the stresses and displacements in a linear viscoelastic body, use is made of the analogy between equations (20) and (21). Namely, the solution for an equivalent elastic problem is obtained first. It can be shown that in many cases the viscoelastic solution is then obtained by replacing the elastic constants by the analogous viscoelastic operators and integrating with respect to time.

The theoretical restriction on this method is that the time entering into the elastic solution must be of a special type; that is, the boundary conditions must be of the proportional loading type and there can be no stress waves. Proportional loading means that the space and time dependence in the non-zero prescribed boundary conditions appear as separate factors. A long, hollow cylinder with only a uniform, time dependent pressure on the internal surface would be a particularly simple example of this. In this case, the only non-zero boundary condition is on the radial stress  $-\sigma_r = p_i f(t)$ , in which the time dependence is given by the factor  $f(t)$ .

From a practical standpoint of being able to perform the integration without excessive difficulty, further restrictions are imposed. In particular, the operators should be simple, as in (9), and the material constants in the elastic solution should not appear in a complicated fashion.

If the elastic problem does not contain these simplifications, then Laplace transform techniques should be used.



By this method, the time variable can be integrated out of the equations involved in the formulation to yield equations which are analogous to those in the elastic problem. This "associated elastic problem" is then solved. Next the resulting expressions are subjected to an inverse Laplace transform to obtain the final time and space dependent stresses and displacements. This method will be further discussed and applied to solid propellant grains in other reports.

It can be shown that when a three-dimensional elastic body is homogeneous and isotropic there can exist no more than two independent elastic constants<sup>(3)</sup> (engineers generally use Young's modulus  $E$  and Poisson's ratio  $\nu$ ). Thus, there are two independent relations between  $E$ ,  $\nu$ ,  $G$  and  $K$  which are:

$$E = \frac{9KG}{3K+G} \quad (22)$$

$$\nu = \frac{3K-2G}{2(3K+G)} \quad (23)$$

Similarly, only two independent operator relations can exist for a homogeneous, isotropic, linear viscoelastic body. Consequently, if  $G$  and  $K$  appear in an elastic solution, we obtain the solution for the viscoelastic body by letting

$$G \rightarrow \frac{Q}{P} \quad (24)$$

$$K \rightarrow \frac{Q'}{P'} \quad (25)$$

and then integrating with respect to time. If  $\nu$  and  $E$  appear, which is usually the case, then the appropriate substitution for  $E$  is

$$E \rightarrow \frac{Q''}{P''} \quad (26)$$

or

$$E \rightarrow \frac{9QQ'}{3Q'P+QP'} \quad (27)$$

and for  $\nu$

$$\nu \rightarrow \frac{1}{2} \frac{Q''}{P''} \frac{P}{Q} - 1 \quad (28)$$

or

$$\nu \rightarrow \frac{3Q'P-2QP'}{2(3Q'P+QP')} \quad (29)$$

The choice of using either (26) or (27) for  $E$  and (28) or (29) for  $\nu$  will depend on which operators are known from experiments.

For low rate loading of many rubbers and plastics above a given temperature, the shear or distortion strains are generally much larger than the extensional strains which contribute to volume change. From equation (15) it is seen that a reasonable assumption for this case would be

$$P' = 0 \quad (\text{infinite bulk modulus})$$

which reduces (27) and (29) to

$$E \rightarrow 3 \frac{Q}{P} \quad (30)$$

$$\nu \rightarrow \frac{1}{2} \quad (31)$$

#### A Simple Example

In order to clarify the procedure of solving a viscoelastic stress problem, a simple example based upon relation (9) discussed earlier will be shown, in which

$$\frac{Q''}{P} = \eta_v \frac{\partial}{\partial t} + E_v \quad (32)$$

$$\nu = \frac{1}{2} \quad (33)$$

(The partial derivative notation is used now since the stresses and displacements may be functions of both time and position). Consider the simple problem of a very long, un-bonded thick walled cylinder that is internally pressurized as shown in Figure 3.

The radial stress in a similar elastic cylinder is<sup>(4)</sup>

$$\sigma_r = \frac{a^2 p_i}{b^2 - a^2} \left( 1 - \frac{b^2}{r^2} \right) \quad (34)$$



It is seen that no material constants appear in (34). The radial stress in a viscoelastic cylinder is therefore identical to that in an elastic cylinder. Thus, we arrive at an important fact: elastic and linear viscoelastic stresses are identical if the elastic solution does not contain material constants.

By integration of the strain-displacement relations, the radial displacement for the elastic body is found to be

$$u = p_i r \left( \frac{1+\nu}{E} \right) \left\{ \left[ \frac{a^2}{b^2 - a^2} \right] \left[ \left( \frac{b}{r} \right)^2 + (1 - 2\nu) \left( \frac{b}{a} \right)^2 \right] - (1 - 2\nu) \right\} \quad (35)$$

The viscoelastic situation is obtained by letting

$$E \rightarrow \eta_v \frac{\partial}{\partial t} + E_v \quad (36)$$

$$\text{and } \nu \rightarrow \frac{1}{2} \quad (37)$$

which yields:

$$\left( \eta_v \frac{\partial}{\partial t} + E_v \right) u = \frac{3}{2} \frac{p_i}{r} \left( \frac{a^2 b^2}{b^2 - a^2} \right) \quad (38)$$

If  $p_i$  for example is a constant pressure applied instantaneously when  $t = 0$ , equation (38) is readily integrated to obtain the time dependent radial viscoelastic displacement,

$$u = \frac{3}{2} \frac{p_i}{E_v r} \left( \frac{a^2 b^2}{b^2 - a^2} \right) \left( 1 - e^{-\frac{t}{\tau_v}} \right) \quad (39)$$

Note that when  $t \rightarrow 0$ , then  $u \rightarrow 0$ ; and when  $t \rightarrow \infty$ ,  $u$  corresponds to the displacement of an elastic cylinder with Young's modulus  $E_v$  and Poisson's ratio  $\nu = \frac{1}{2}$ .

### Restrictions and Applications

These introductory remarks on viscoelasticity point out the close correspondence that exists in certain cases between the linear viscoelastic

stress problem and an equivalent elastic problem. Thus, the need for having elastic solutions in order to compute the viscoelastic stresses is indeed evident. The important assumption needed to solve these problems in the above fashion is that the strains must be small. This stems from both material and mathematical considerations. The first one has already been mentioned in that plastics and rubbers exhibit linear viscoelastic behavior only for small strains. The other is due to the fact that the mathematics in the ordinary theory of elasticity is developed on the assumption of small strains. However, it should not be felt that application of the theory is seriously restricted. For one thing, when a propellant grain is rapidly loaded, rupture may occur at very small strains. This is also true at low temperatures. Another important use of the theory is for the indication of trends as strains become large; in fact the theory may still give fair accuracy for strains of several percent.

We turn now to the relation between the mechanical behavior of a linear viscoelastic material and the operator equation used in describing it.

### Model Representation

The operator equations (21), which define the stress-strain behavior of a linear viscoelastic material, can be represented diagrammatically by mechanical models that exhibit the same macroscopic behavior<sup>(5)</sup>. It is important to note that these models will give a description only of the phenomenological behavior of a material, and usually tell nothing of the complex molecular processes causing this behavior. However, they are a convenient means for constructing an operator equation in order to approximate actual viscoelastic behavior.

A simple example of a model is the representation of Hooke's law by a spring as shown in Figure 4a. If we denote the spring constant by a modulus  $m$ , the applied force by stress  $\sigma$ , and the extension by strain  $\epsilon$ , we have a model that evinces the same response as described by any one of the equations (20);  $\sigma$  and  $\epsilon$  are now to be understood to represent either tension, shear, or bulk behavior.



It will be shown in the following analysis that linear viscoelastic action can be represented by models made up of Hookean spring elements along with dashpots that exhibit Newtonian flow behavior (stress proportional to strain rate). The dashpot element is represented schematically in Figure 4b, in which the proportionality constant is defined as the viscosity  $\eta$ .

Now, since the response in shear, tension, and bulk is assumed to be linearly viscoelastic, the stress  $\sigma$  and strain  $\epsilon$  used with the models will be usually assumed, for convenience, to represent any one of these three types of loading. However, there is an exception to this rule for certain models. Specifically, when a simple shear stress is applied to a noncross-linked polymer element, its deformation increases indefinitely. However if this same polymer is subjected to a hydrostatic pressure the volume cannot decrease in an unlimited manner, but must approach an equilibrium value. Consequently, a model that evinces unlimited flow behavior cannot be used to define an operator equation relating hydrostatic pressure (stress) and relative volume change (strain). In contrast, a cross-linked polymer specimen subjected to a shear stress will always reach an equilibrium deformation. We therefore have the rule that in describing tension, bulk, and shear response for a cross-linked polymer, or bulk response for a noncross-linked polymer, a model should be used that does not exhibit unlimited flow under stress.

### Voigt Model

One of the simplest models exhibiting viscoelastic behavior is the Voigt model. It consists of a spring and dashpot in parallel as shown in Figure 5a.

The procedure used for obtaining the operator equation for a particular model is: (1) write an equation of stress equilibrium in which the applied stress is balanced by the internal stresses on the elements, as shown in Figure 5a; (2) relate the overall strain (or extension)  $\epsilon$  of the element to the internal strains. For the Voigt model this step is trivial since the overall strain is the same as in the dashpot and spring. It is seen that equation (8), discussed previously, is represented by this model.

The behavior in standard tests is shown in Figures 5b and 5c. In a creep test, the equation for strain is found by integrating the operator equation in which the applied stress is constant. The initial condition needed to determine the constant of integration is  $\epsilon = 0$  when  $t = 0$ . It should be noted that there is no instantaneous strain, whereas an actual propellant does deform immediately (neglecting inertia effects). The recovery equation of strain is obtained by integrating the operator equation with the stress set equal to zero, and the initial condition  $\epsilon = \epsilon_1$ , when  $t = t_1$ . The curve shows that the model completely recovers to its original length as  $t \rightarrow \infty$ .

The reason for defining previously the ratio  $\eta_v/m_v$  as retardation time  $\tau_v$ , is seen by the creep behavior. Here,  $\tau_v$  represents a quantity with the dimension of time and has the effect of shifting the time scale in regards to the delayed action of the material. For example, when  $\tau_v$  is large, the strain is retarded so that it increases slowly; whereas if  $\tau_v$  is small, the curve shifts to the left and the equilibrium strain is approached quite rapidly with very little retardation.

When this model is held at a constant strain, as in a stress relaxation test, the stress remains constant since no flow occurs in the dashpot. Thus, the Voigt model does not exhibit stress relaxation, while a propellant does.

With a constant strain rate, the stress immediately jumps to a finite value and then increases linearly. This behavior also is not representative of a propellant since the stress in a real material increases continuously from zero at a decreasing rate.

### Maxwell Model

The Maxwell model in Figure 6a is another basic model exhibiting viscoelastic response. The first step in obtaining the operator equation is trivial in this case since the stress in the dashpot is the same as that in the spring. Strains are matched by equating the overall strain to the strain in the dashpot plus that in the spring.

The integrated expression for strain in a creep test (Figure 6b) shows that deformation occurs immediately, and that unlimited flow

occurs - i. e., the strain does not approach an upper limit. When the stress is removed, there is an immediate recovery to a finite strain which remains as a permanent deformation.

This model shows stress relaxation behavior in a manner similar to uncross-linked polymers, in that the stress decays to zero at a decreasing rate.

For this model, the time constant is defined as

$$\tau_m = \frac{\eta_m}{M_m}$$

in which  $\tau_m$  is called the relaxation time. It has the dimension of time and represents the time it takes for the stress to fall to  $(1/e)$  of its original value in a relaxation test. It is, therefore, a measure of the rate at which the relaxation occurs; for example, when the relaxation time  $\tau_m$  is very small, the stress decays to zero almost immediately.

If the model is strained at a constant rate, the stress-time curve in Figure 6d shows that it behaves in a manner similar to a propellant.

#### Generalization of Models

The Voigt and Maxwell models cannot be used in themselves to represent propellant behavior, as was pointed out for a few cases. Therefore, it is necessary to form combinations of these basic units in order to approximate actual linear viscoelastic response. This is usually done by adding Voigt models in series or Maxwell models in parallel to form an array of springs and dashpots. The mathematical relations developed for these models will serve as a useful guide in designing experiments for the testing of propellants.

Subsequent reports will deal with this aspect of the analysis, as well as outlining the general procedure to be used in investigating failure and failure criteria.



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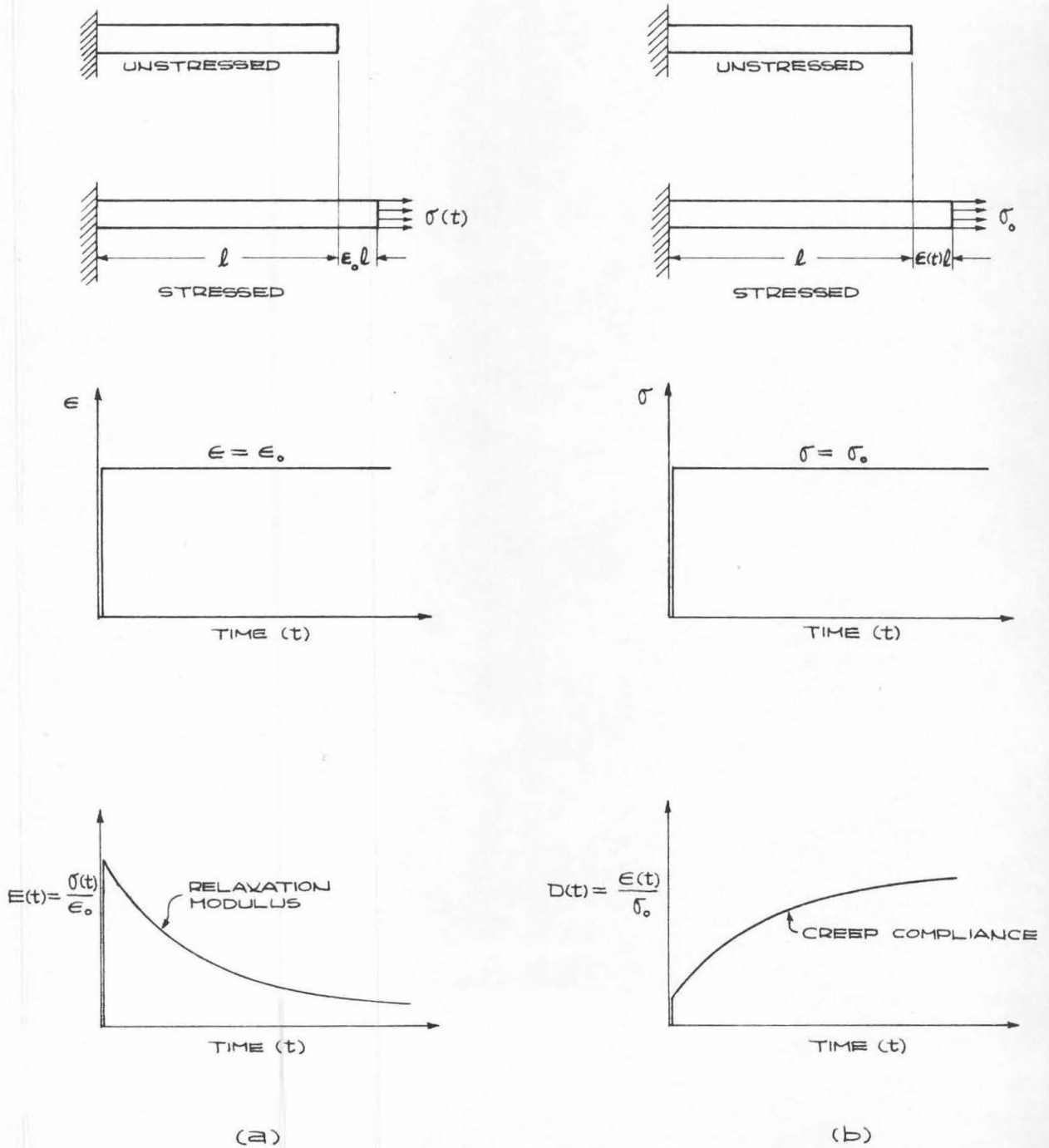


FIG. 1. TYPICAL PROPELLANT BEHAVIOR.

- (a) RELAXATION TEST  
 (b) CREEP TEST

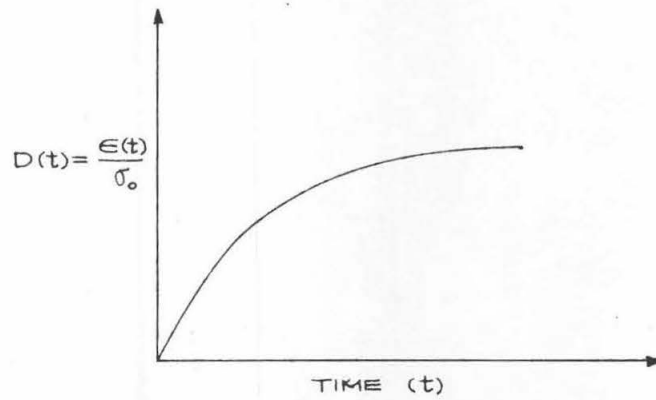


FIG. 2. CREEP COMPLIANCE FOR STRESS STRAIN

LAW:  $\sigma(t) = \left[ E_v + \eta_v \frac{d}{dt} \right] \epsilon(t)$

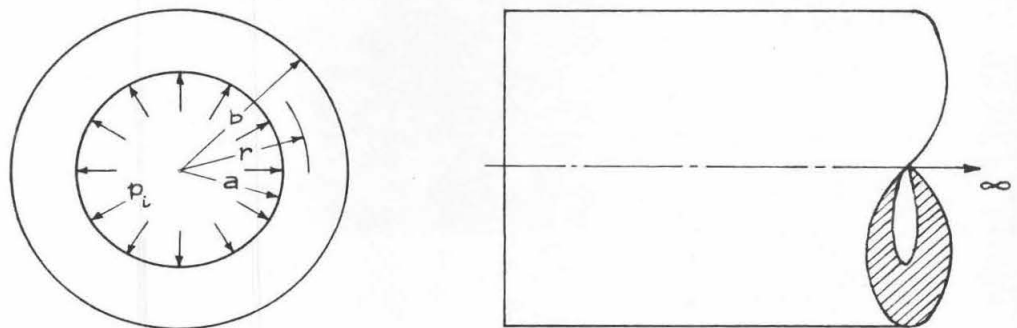
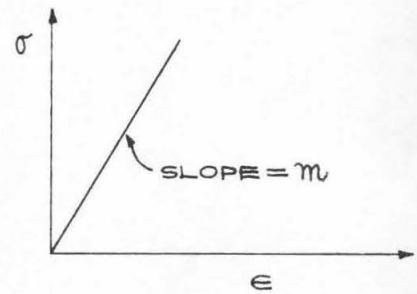
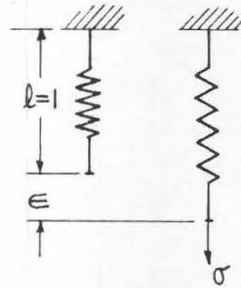


FIG. 3. VERY LONG INTERNALLY PRESSURIZED CYLINDER

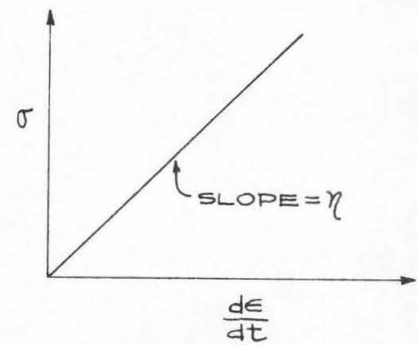
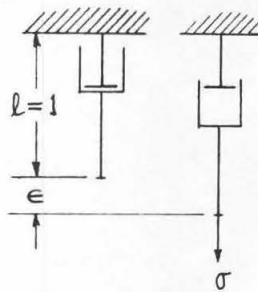


$$\sigma = m \epsilon$$



(a) HOOKEAN SPRING

$$\sigma = \eta \frac{d\epsilon}{dt}$$



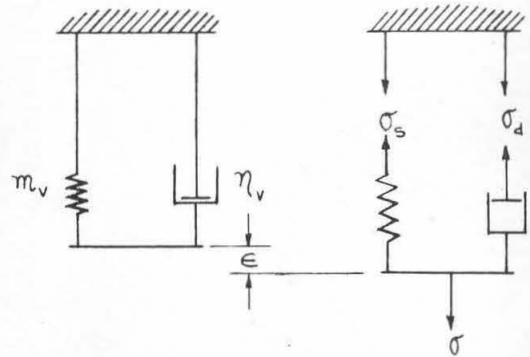
(b) NEWTONIAN DASHPOT

FIG. 4. ONE ELEMENT MODELS

$$\sigma = \sigma_d + \sigma_s$$

WHERE  $\sigma_d = \eta_v \frac{d\epsilon}{dt}$

$$\sigma_s = m_v \epsilon$$



OPERATOR EQUATION:

$$\sigma(t) = \left( \eta_v \frac{d}{dt} + m_v \right) \epsilon(t)$$

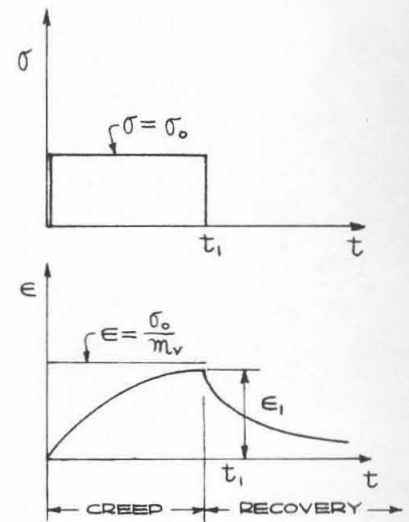
(a) MODEL

$$\sigma = \sigma_0; \quad \epsilon(t) = \frac{\sigma_0}{m_v} \left( 1 - e^{-\frac{t}{\tau_v}} \right); \quad \text{FOR } 0 < t \leq t_1$$

$$\sigma = 0; \quad \epsilon(t) = \epsilon_1 e^{-\frac{(t-t_1)}{\tau_v}}; \quad \text{FOR } t > t_1$$

WHERE  $\tau_v = \frac{\eta_v}{m_v}$

(b) CREEP & RECOVERY



$$\epsilon = R t \quad (R = \text{STRAIN RATE})$$

$$\sigma = R m_v [t + \tau_v]$$

(c) CONSTANT STRAIN RATE

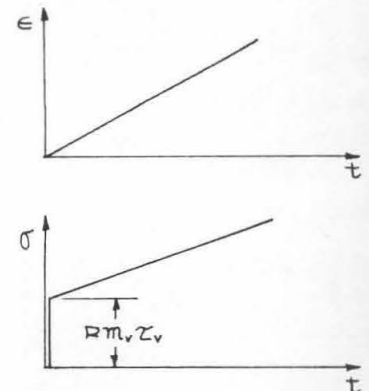


FIG. 5. TWO ELEMENT VOIGT MODEL

$$\epsilon = \epsilon_s + \epsilon_d$$

$$\text{OR } \frac{d\epsilon}{dt} = \frac{d\epsilon_s}{dt} + \frac{d\epsilon_d}{dt}$$

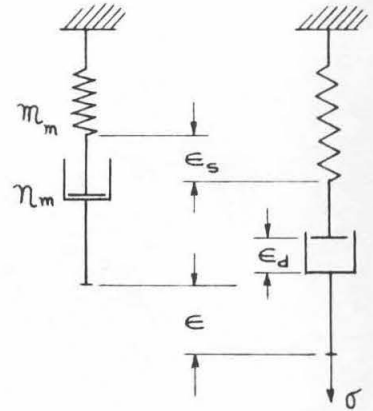
$$\text{WHERE } \frac{d\epsilon_s}{dt} = \frac{1}{m_m} \frac{d\sigma}{dt}$$

$$\frac{d\epsilon_d}{dt} = \frac{1}{n_m} \sigma$$

OPERATOR EQUATION:

$$\left( \frac{1}{m_m} \frac{d}{dt} + \frac{1}{n_m} \right) \sigma(t) = \frac{d\epsilon(t)}{dt}$$

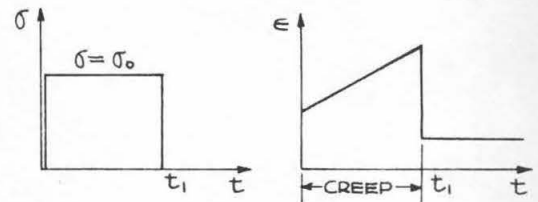
(a) MODEL



$$\sigma = \sigma_0; \epsilon(t) = \frac{\sigma_0}{m_m} \left( 1 + \frac{t}{Z_m} \right); \text{ FOR } 0 < t \leq t_1$$

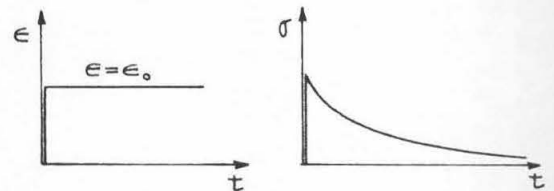
$$\text{WHERE } Z_m = \frac{n_m}{m_m}$$

$$\sigma = 0; \epsilon(t) = \frac{\sigma_0}{m_m} \frac{t_1}{Z_m}; \text{ FOR } t > t_1$$



(b) CREEP

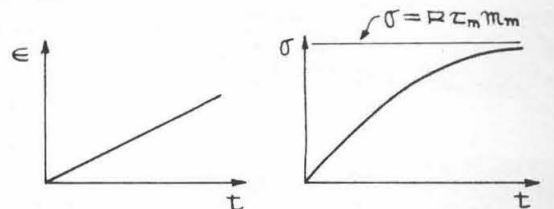
$$\epsilon = \epsilon_0; \sigma(t) = \epsilon_0 m_m e^{-\frac{t}{Z_m}}; \text{ FOR } t > 0$$



(c) RELAXATION

$$\epsilon = Rt \quad (R = \text{STRAIN RATE})$$

$$\sigma = R Z_m m_m (1 - e^{-\frac{t}{Z_m}})$$



(d) CONSTANT STRAIN RATE

FIG. 6. TWO ELEMENT MAXWELL MODEL



## II. ELASTIC SOLUTIONS FOR CYLINDERS

This section of the report consists of a compilation of various elastic solutions pertaining to right circular cylinders under various loading conditions of tension, pressure and torsion. It will serve as a convenient collection of solutions which are of direct application to the analysis of solid propellant viscoelastic behavior using the analogy discussed in the first section. The viscoelastic solutions to be derived from these elastic solutions can eventually be compared to the results from a variety of tests on actual or simulated solid propellant configurations. Further extension to star geometries is anticipated through the use of stress-concentration factors.

These elastic solutions will be amplified to include various temperature loading conditions. Inasmuch as there are many standard texts readily available which relate to the theory of elasticity, derivations of the following results are omitted.

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## DEFINITIONS OF SYMBOLS AND TERMS

Bonded	- adhesion of propellant to case
E	- Young's modulus
End effects	- effects due to finite length cylinders: $z = 0$ or $z = l$
Necking effect	- shrinking of cross-section due to elongation
Plane stress	- no stress in axial direction: $\sigma_z = 0$
Plane strain	- no strain (displacement) in axial direction: $\epsilon_x = 0$
R	- body force in radial direction as, for example, centrifugal force
r	- radial coordinate
Second order	- a smaller quantity which is proportional to square of the variable
	- most elastic solutions are first order linear theory
Small strains	- strains of the order of a few percent for linear theory
St. Venant	- (a) Localized effects at boundaries die out as boundaries are remote
	- (b) Simple first order torsion in which ends rotate as solid disks
Subscript ( $\begin{smallmatrix} r \\ \theta \end{smallmatrix}$ )	- used to indicate two solutions simultaneously - upper sign option with upper subscript, etc.
u	- radial displacement
v	- tangential displacement (in $\theta$ direction)
w	- axial displacement
Z	- body force in axial direction, e.g., gravity
z	- axial coordinate
$\gamma$	- shear strain
$\nabla$	- del operator for differentiation - see page i for definition of $\nabla^2$
$\epsilon$	- strain
$\textcircled{H}$	- body force in the tangential (or $\theta$ ) direction
$\theta$	- tangential coordinate of the cylindrical (or polar) system
$\mu$	- shear modulus of elasticity - see page i
$\nu$	- Poisson's ratio
$\sigma$	- stress
$\tau$	- shear stress
$\phi$	- general stress function

## CYLINDRICAL EQUATIONS:

EQUILIBRIUM :

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0$$

$$\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2 \tau_{r\theta}}{r} + \theta = 0$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + Z = 0$$

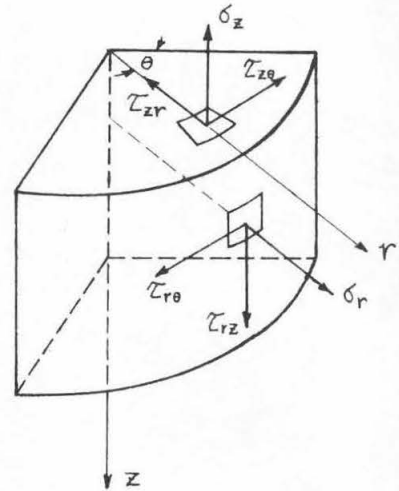
STRESS - STRAIN :

$$\epsilon_r = \frac{1}{E} [\sigma_r - \nu (\sigma_\theta + \sigma_z)]$$

$$\epsilon_\theta = \frac{1}{E} [\sigma_\theta - \nu (\sigma_r + \sigma_z)]$$

$$\epsilon_z = \frac{1}{E} [\sigma_z - \nu (\sigma_r + \sigma_\theta)]$$

$$\gamma_{r\theta} = \frac{\tau_{r\theta}}{\mu}, \quad \gamma_{rz} = \frac{\tau_{rz}}{\mu}, \quad \gamma_{\theta z} = \frac{\tau_{\theta z}}{\mu}, \quad \mu = \frac{E}{2(1+\nu)}$$



DISPLACEMENT :

$$\epsilon_r = \frac{\partial u}{\partial r}, \quad \epsilon_\theta = \frac{u}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \epsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{r\theta} = \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \quad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \quad \gamma_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta}$$

COMPATIBILITY :

$$\nabla^4 \phi = 0, \quad \phi = \text{stress function}$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$



## AXIALLY SYMMETRIC EQUATIONS

EQUILIBRIUM:

$$\left. \begin{aligned} \frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + \rho &= 0 \\ \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + \gamma &= 0 \end{aligned} \right\} \text{ No dependence on } \theta, \tau_{r\theta} = \tau_{\theta z} = 0$$

STRAIN - DISPLACEMENT:

$$\epsilon_r = \frac{\partial u}{\partial r}, \quad \epsilon_\theta = \frac{u}{r}, \quad \epsilon_z = \frac{\partial w}{\partial z}, \quad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

(stress - strain relation unchanged)

STRESS FUNCTIONAL APPROACH (NO BODY FORCES):

$$\nabla^4 \phi = 0, \quad \nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

$$\sigma_r = \frac{\partial}{\partial z} \left[ \nu \nabla^2 \phi - \frac{\partial^2 \phi}{\partial r^2} \right]$$

$$\sigma_\theta = \frac{\partial}{\partial z} \left[ \nu \nabla^2 \phi - \frac{1}{r} \frac{\partial \phi}{\partial r} \right]$$

$$\sigma_z = \frac{\partial}{\partial z} \left[ (2 - \nu) \nabla^2 \phi - \frac{\partial^2 \phi}{\partial z^2} \right]$$

$$\tau_{rz} = \frac{\partial}{\partial r} \left[ (1 - \nu) \nabla^2 \phi - \frac{\partial^2 \phi}{\partial z^2} \right]$$

$$\epsilon_r = - \frac{(1 + \nu)}{E} \frac{\partial^3 \phi}{\partial r^2 \partial z}$$

$$\epsilon_\theta = - \frac{(1 + \nu)}{E} \frac{1}{r} \frac{\partial^2 \phi}{\partial r \partial z}$$

$$\epsilon_z = \frac{(1 + \nu)}{E} \frac{\partial}{\partial z} \left[ (1 - 2\nu) \nabla^2 \phi + \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right]$$

$$u = - \frac{(1 + \nu)}{E} \frac{\partial^2 \phi}{\partial r \partial z}$$

$$w = \frac{(1 + \nu)}{E} \left[ (1 - 2\nu) \nabla^2 \phi + \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right]$$

AXIALLY SYMMETRIC EQUATIONS - TWO DIMENSIONAL:

EQUILIBRIUM:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} + Q = 0$$

STRAIN - DISPLACEMENT:

$$\epsilon_r = \frac{\partial u}{\partial r}, \quad \epsilon_\theta = \frac{u}{r}, \quad \epsilon_z = \frac{\partial w}{\partial z}$$

PLANE STRESS:

$$\sigma_z = 0$$

$$\epsilon_r = \frac{1}{E} (\sigma_r - \nu \sigma_\theta)$$

$$\epsilon_\theta = \frac{1}{E} (\sigma_\theta - \nu \sigma_r)$$

$$\epsilon_z = -\frac{\nu}{E} (\sigma_r + \sigma_\theta)$$

PLANE STRAIN:

$$\epsilon_z = 0$$

$$\epsilon_r = \frac{1}{E} [\sigma_r - \nu (\sigma_\theta + \sigma_z)] = \left( \frac{1-\nu^2}{E} \right) \left[ \sigma_r - \left( \frac{\nu}{1-\nu} \right) \sigma_\theta \right]$$

$$\epsilon_\theta = \frac{1}{E} [\sigma_\theta - \nu (\sigma_r + \sigma_z)] = \left( \frac{1-\nu^2}{E} \right) \left[ \sigma_\theta - \left( \frac{\nu}{1-\nu} \right) \sigma_r \right]$$

Hence, to change plane stress into plane strain, substitute:

$$\frac{E}{1-\nu^2} \text{ for } E \text{ and } \frac{\nu}{1-\nu} \text{ for } \nu.$$

STRESS FUNCTIONAL APPROACH (NO BODY FORCE):

$$\nabla^4 \phi = 0$$

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}$$

$$\sigma_r = \frac{1}{r} \frac{\partial \phi}{\partial r} = \frac{A}{r^2} + 2C$$

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2} = -\frac{A}{r^2} + 2C$$

$$\phi = A \ln r + C r^2$$

TENSION - SOLID OR HOLLOW CYLINDER - NO CASE

BOUNDARY CONDITIONS:

$$\left. \begin{aligned} \sigma_z = \sigma_o = \frac{P}{A} = \frac{P}{\pi(b^2 - a^2)} : z=0, l \\ \sigma_r = 0 : r=a, b \end{aligned} \right\} \begin{array}{l} \text{for solid cylinder} \\ \text{let } a \rightarrow 0 \end{array}$$

STRAINS:

$$\epsilon_r = \epsilon_\theta = -\frac{\nu \sigma_o}{E}$$

$$\epsilon_z = \frac{\sigma_o}{E}$$

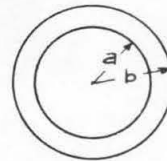
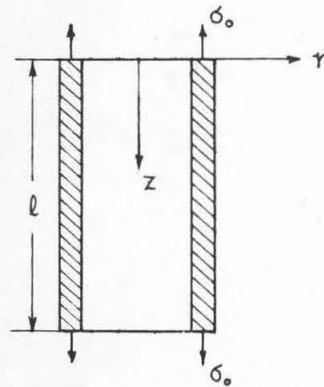
DISPLACEMENTS:

$$u = -\frac{\nu \sigma_o}{E} r$$

$$w = \frac{\sigma_o}{E} z$$

ASSUMPTIONS:

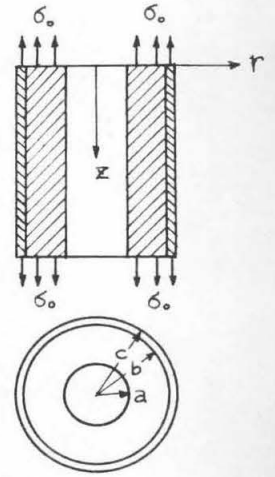
1. No end effects - either ends contract as given by  $u$  or we consider only regions  $> 0(b-a)$  from ends (St. Venant).
2.  $\sigma_o$  distributed uniformly over ends, or equivalent with same total pressure & consider only regions  $> 0(b-a)$  from ends.
3. Small strains where  $A$  can be considered constant (no necking effect).



# TENSION - SOLID OR HOLLOW CYLINDER - WITH CASE - BONDED-CYLINDER AND CASE IN TENSION-UNIFORM STRESS

BOUNDARY CONDITIONS:

$$\left. \begin{aligned} \sigma_z = \sigma_o = \frac{P}{\pi(c^2 - a^2)}, \quad z = 0, \quad l \\ \sigma_r = 0: \quad r = a, \quad c \\ u = u_c: \quad r = b \end{aligned} \right\} \begin{array}{l} \text{For solid} \\ \text{cylinder;} \\ \text{let } a \rightarrow 0 \\ \text{throughout} \end{array}$$



$$c - b = h$$

STRESS ON CYLINDER:

$$\sigma_z = \sigma_o$$

$$\sigma_r = -p': \quad r = b,$$

$$p' = - \frac{\nu \left(1 - \frac{\nu_c E}{\nu E_c}\right) \sigma_o}{\frac{b^2}{b^2 - a^2} \left[ (1 - \nu) + (1 + \nu) \frac{a^2}{b^2} \right] + \frac{b^2}{c^2 - b^2} \left( \frac{E}{E_c} \right) \left[ (1 - \nu_c) + (1 + \nu_c) \frac{c^2}{b^2} \right]}$$

FOR THIN CASE:

$$p' = - \frac{\nu \left(1 - \frac{\nu_c E}{\nu E_c}\right) \sigma_o}{\frac{b^2}{b^2 - a^2} \left[ (1 - \nu) + (1 + \nu) \frac{a^2}{b^2} \right] + \frac{b E}{h E_c}}$$

STRAINS & DISPLACEMENTS OF CYLINDER:

$$\epsilon_r = - \frac{1}{E} \left\{ \nu \sigma_o + \frac{b^2 p'}{b^2 - a^2} \left[ (1 - \nu) + (1 + \nu) \frac{a^2}{r^2} \right] \right\}$$

$$\epsilon_z = \frac{\sigma_o}{E}$$

$$u = - \frac{1}{E} \left\{ \nu \sigma_o r + \frac{b^2 p'}{b^2 - a^2} \left[ (1 - \nu) r + (1 + \nu) \frac{a^2}{r} \right] \right\},$$

$$w = + \frac{\sigma_o}{E} z$$

STRESSES ON CASE:

$$\sigma_z = \sigma_o$$

$$\sigma_r = \frac{b^2 p'}{c^2 - b^2} \left( 1 - \frac{c^2}{r^2} \right)$$

STRAINS & DISPLACEMENTS OF CASE:

$$\epsilon_r = -\frac{1}{E_c} \left\{ \nu_c \sigma_o - \frac{b^2 p'}{c^2 - b^2} \left[ (1 - \nu_c) - (1 - \nu_c) \frac{c^2}{r^2} \right] \right\}$$

$$\epsilon_\theta = -\frac{1}{E_c} \left\{ \nu_c \sigma_o - \frac{b^2 p'}{c^2 - b^2} \left[ (1 - \nu_c) + (1 + \nu_c) \frac{c^2}{r^2} \right] \right\}$$

$$\epsilon_z = \frac{2\nu_c}{E_c} \left[ \nu_c \sigma_o - \frac{b^2 p'}{c^2 - b^2} (1 - \nu_c) \right]$$

$$u_c = -\frac{1}{E_c} \left\{ \nu_c \sigma_o r - \frac{b^2 p'}{c^2 - b^2} \left[ (1 - \nu_c) r + (1 + \nu_c) \frac{c^2}{r} \right] \right\}$$

$$w_c = \frac{2\nu_c}{E_c} \left[ \nu_c \sigma_o - \frac{b^2 p'}{c^2 - b^2} (1 - \nu_c) \right] z$$

ASSUMPTIONS :

1. No end effects
2. Small strains
3. Shear due to  $w - w_c$  of lesser importance than matching  $u = u_c$

For  $[w_c = w]_{a=0}$ :

$$p' \Big|_{a=0} = \frac{-\frac{1}{E} (1 - 2\nu_c^2 \frac{E}{E_c}) \sigma_o}{\frac{b^2}{c^2 - b^2} (1 - \nu_c)} \approx -\frac{\frac{1}{E} (1 - 2\nu_c^2 \frac{E}{E_c}) \sigma_o}{\frac{b}{2h} (1 - \nu_c)}$$

$$\approx 0 \left( \frac{h}{bE} \sigma_o \right) \ll \sigma_o$$



TENSION - SOLID OR HOLLOW CYLINDER - WITH CASE - BONDED CYLINDER & CASE IN TENSION - UNIFORM STRAIN

BOUNDARY CONDITIONS:

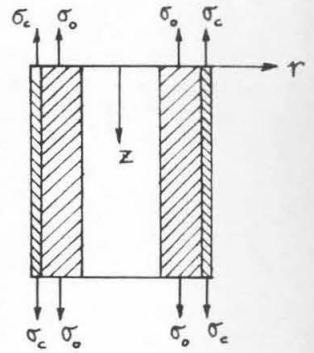
$$\sigma_r = 0: \quad r = a, c$$

$$u = u_c: \quad r = b$$

$$w = w_c: \quad z = 0, l$$

$$\bar{\sigma} = \frac{P}{A} = \frac{\pi [(b^2 - a^2) \sigma_o + (c^2 - b^2) \sigma_c]}{\pi (c^2 - a^2)}$$

$$\approx \frac{(b^2 - a^2) \sigma_o + 2bh \sigma_c}{b^2 - a^2} \quad [\text{Thin Case}]$$



STRESSES ON CYLINDER:

$$\sigma_z = \sigma_o$$

$$\sigma_r = -p': \quad r = b$$

$$p' = - \frac{\nu \sigma_o - \nu_c \frac{E}{E_c} \sigma_c}{\frac{b^2}{b^2 - a^2} \left[ (1 - \nu) + (1 + \nu) \frac{a^2}{b^2} \right] + \frac{b^2}{c^2 - a^2} \left( \frac{E}{E_c} \right) \left[ (1 - \nu_c) + (1 + \nu_c) \frac{c^2}{b^2} \right]}$$

FOR THIN CASE:

$$p' = - \frac{\nu \sigma_o - \nu_c \frac{E}{E_c} \sigma_c}{\frac{b^2}{b^2 - a^2} \left[ (1 - \nu) + (1 + \nu) \frac{a^2}{b^2} \right] + \frac{b E}{h E_c}}$$

STRAINS & DISPLACEMENTS OF CYLINDER:

$$\epsilon_{(r)} = - \frac{1}{E} \left\{ \nu \sigma_o + \frac{b^2 p'}{b^2 - a^2} \left[ (1 - \nu) + (1 + \nu) \frac{a^2}{r^2} \right] \right\}$$

$$\epsilon_z = \frac{\sigma_o}{E}$$

$$u = - \frac{1}{E} \left\{ \nu \sigma_o r + \frac{b^2 p'}{b^2 - a^2} \left[ (1 - \nu) r + (1 + \nu) \frac{a^2}{r} \right] \right\}$$

$$w = \frac{\sigma_o}{E} z$$

STRESSES ON CASE :

$$\sigma_z = \sigma_c$$

$$\sigma_r = \frac{b^2 p'}{c^2 - b^2} \left(1 - \frac{c^2}{r^2}\right)$$

STRAINS &amp; DISPLACEMENTS OF CASE :

$$\epsilon_{(r)} = -\frac{1}{E_c} \left\{ \nu_c \sigma_c - \frac{b^2 p'}{c^2 - b^2} \left[ (1 - \nu_c) + (1 + \nu_c) \frac{c^2}{r^2} \right] \right\}$$

$$\epsilon_z = \frac{2\nu_c}{E_c} \left[ \nu_c \sigma_c - \frac{b^2 p'}{c^2 - b^2} (1 - \nu_c) \right]$$

$$u_c = -\frac{1}{E_c} \left\{ \nu_c \sigma_c r - \frac{b^2 p'}{c^2 - b^2} \left[ (1 - \nu_c) r + (1 + \nu_c) \frac{c^2}{r} \right] \right\}$$

$$w_c = \frac{2\nu_c}{E_c} \left[ \nu_c \sigma_c - \frac{b^2 p'}{c^2 - b^2} (1 - \nu_c) \right] z$$

ASSUMPTIONS :

1. No end effects
2. Small strains

$$3. w = w_c : \quad \sigma_o = \frac{2\nu_c E}{E_c} \left[ \nu_c \sigma_c - \frac{b p'}{c^2 - b^2} (1 - \nu_c) \right]$$

TENSION - SOLID OR HOLLOW CYLINDER - WITH CASE - BONDED  
CASE IN TENSION OR CYLINDER IN TENSION

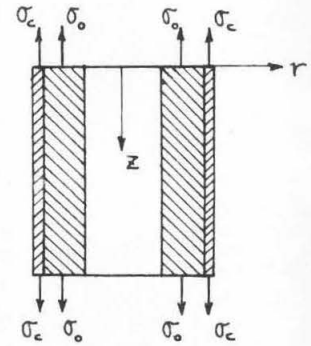
See previous pages for cylinder & case both in tension and uniform strain :

FOR CASE ONLY IN TENSION :

Let :  $\sigma_c = 0$  &  $w \neq w_c$

FOR CYLINDER ONLY IN TENSION :

Let :  $\sigma_c = 0$  &  $w \neq w_c$



ASSUMPTIONS :

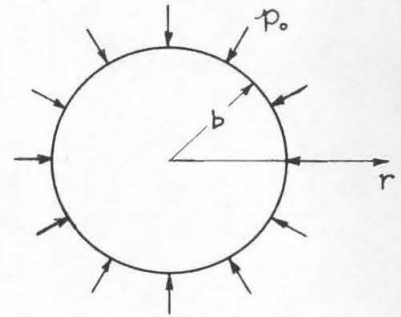
1. No end effects.
2. Small strains.
3. Shear due to  $w - w_c$  of lesser importance than matching  $u = u_c$ .

PRESSURE - SOLID CYLINDER - NO CASETWO DIMENSIONAL PLANE STRESS:

BOUNDARY CONDITIONS:

$$\sigma_r = -p_0 : r = b$$

$$\sigma_z = 0 : z = 0, l$$



STRESSES:

$$\sigma_r = \sigma_\theta = -p_0$$

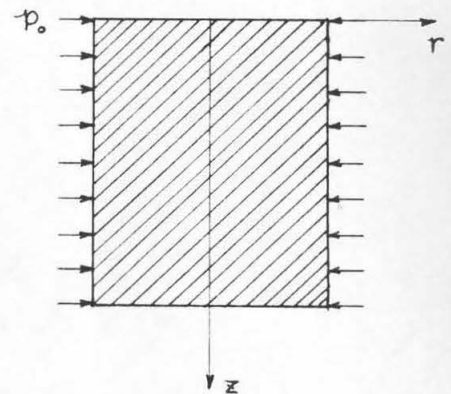
STRAINS &amp; DISPLACEMENTS:

$$\epsilon_r = \epsilon_\theta = -\frac{(1-\nu)}{E} p_0$$

$$u = -\frac{(1-\nu)}{E} p_0 r$$

$$\epsilon_z = \frac{2\nu}{E} p_0$$

$$w = \frac{2\nu}{E} p_0 z$$

TWO DIMENSIONAL PLANE STRAIN:Boundary conditions & stresses same except  $\epsilon_z = 0$ :

$$z = 0, l \quad \text{and} \quad \sigma_z = -2\nu p_0$$

STRAINS &amp; DISPLACEMENTS:

$$\epsilon_r = \epsilon_\theta = -\frac{(1+\nu)(1-2\nu)}{E} p_0$$

$$u = -\frac{(1+\nu)(1-2\nu)}{E} p_0 r$$

ASSUMPTION:

1. No end effects

PRESSURE - SOLID CYLINDER - WITH CASETWO DIMENSIONAL PLANE STRESS:

BOUNDARY CONDITIONS:

$$\sigma_r = -p_0 : r = c$$

$$u = u_c, \quad z = 0 : r = b$$

$$\sigma_z = 0 : z = 0, l$$

STRESSES IN SOLID CYLINDER:

$$\sigma_r = \sigma_\theta = -p' = \frac{-2c^2 p_0}{(1+\nu)c^2 + (1-\nu)b^2 + (1-\nu)(c^2 - b^2) \frac{E_c}{E}}$$

 $p'$  = pressure between cylinder and case

FOR THIN CASE:

$$p' = \frac{p_0}{1 + (1-\nu) \frac{h E_c}{b E}}$$

STRAINS &amp; DISPLACEMENTS IN SOLID CYLINDER:

$$\epsilon_r = \epsilon_\theta = -\frac{(1-\nu)}{E} p'$$

$$u = -\frac{(1-\nu)}{E} p' r$$

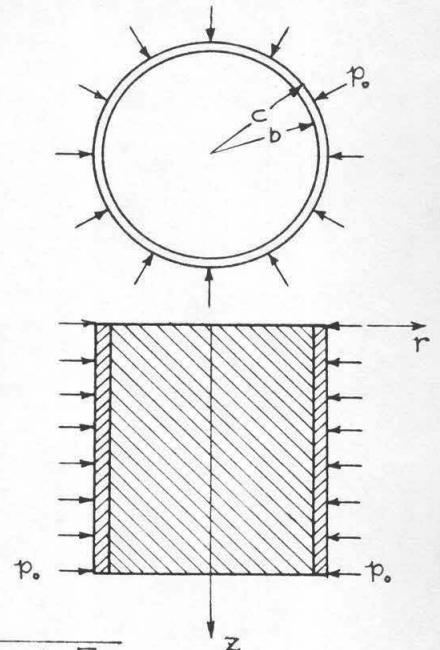
$$\epsilon_z = \frac{2\nu}{E} p'$$

$$w = \frac{2\nu}{E} p' z$$

STRESSES IN CASE:

$$\sigma_r = \frac{b^2 c^2 (p_0 - p')}{c^2 - b^2} \frac{1}{r^2} + \frac{p' b^2 - p_0 c^2}{c^2 - b^2}$$

$$\sigma_\theta = -\frac{b^2 c^2 (p_0 - p')}{c^2 - b^2} \frac{1}{r^2} + \frac{p' b^2 - p_0 c^2}{c^2 - b^2}$$





## STRAINS &amp; DISPLACEMENTS IN CASE

$$\epsilon_r = \frac{1}{(c^2 - b^2)E_c} \left[ \frac{b^2 c^2 (p_o - p')(1 + \nu_c)}{r^2} + (p'b^2 - p_o c^2)(1 - \nu_c) \right]$$

$$\epsilon_\theta = \frac{1}{(c^2 - b^2)E_c} \left[ -\frac{b^2 c^2 (p_o - p')(1 + \nu_c)}{r^2} + (p'b^2 - p_o c^2)(1 - \nu_c) \right]$$

$$\epsilon_z = -\frac{2 \nu_c (p'b^2 - p_o c^2)}{(c^2 - b^2)E_c}$$

$$u = \frac{1}{(c^2 - b^2)E_c} \left[ -\frac{b^2 c^2 (p_o - p')(1 + \nu_c)}{r} + (p'b^2 - p_o c^2)(1 - \nu_c)r \right]$$

$$w = -\frac{2 \nu_c (p'b^2 - p_o c^2)}{(c^2 - b^2)E_c} z$$

## ASSUMPTIONS:

1. No end effects
2. No shear transmitted between cylinder and case.

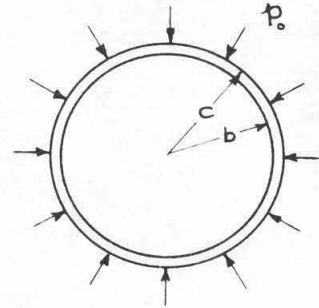
PRESSURE - SOLID CYLINDER - WITH CASE (CONT.)TWO DIMENSIONAL PLANE STRAIN:

BOUNDARY CONDITIONS:

$$\sigma_r = -p_0 : r = c$$

$$u = u_c : r = b$$

$$\epsilon_z = 0 : z = 0, l$$



STRESSES IN SOLID CYLINDER:

$$\sigma_r = \sigma_\theta = -p' = \frac{-2(1-\nu_c)c^2 p_0}{c^2 + (1-2\nu_c)b^2 + (1-2\nu)(c^2 - b^2) \frac{(1+\nu)E_c}{(1+\nu_c)E}}$$

 $p'$  = pressure between cylinder and case

FOR THIN CASE:

$$p' = \frac{p_0}{1 + \frac{(1-2\nu)(1+\nu)}{1-\nu_c^2} \frac{hE_c}{bE}}$$

STRAINS &amp; DISPLACEMENTS IN CYLINDER:

$$\epsilon_r = \epsilon_\theta = - \frac{(1-2\nu)(1+\nu)}{E} p'$$

$$u = - \frac{(1-2\nu)(1+\nu)}{E} p' r$$

STRESSES IN CASE:

$$\sigma_r = \frac{b^2 c^2 (p_0 - p')}{c^2 - b^2} \frac{1}{r^2} + \frac{p' b^2 - p_0 c^2}{c^2 - b^2}$$

$$\sigma_\theta = - \frac{b^2 c^2 (p_0 - p')}{c^2 - b^2} \frac{1}{r^2} + \frac{p' b^2 - p_0 c^2}{c^2 - b^2}$$

$$\sigma_z = \frac{2\nu_c (p' b^2 - p_0 c^2)}{c^2 - b^2}$$

STRAINS & DISPLACEMENTS IN CASE:

$$\epsilon_r = \frac{1+\nu_c}{(c^2-b^2)E_c} \left[ \frac{b^2 c^2 (p_0 - p')}{r^2} + (p' b^2 - p_0 c^2)(1-2\nu_c) \right]$$

$$\epsilon_\theta = \frac{1+\nu_c}{(c^2-b^2)E_c} \left[ -\frac{b^2 c^2 (p_0 - p')}{r^2} + (p' b^2 - p_0 c^2)(1-2\nu_c) \right]$$

$$u = \frac{1+\nu_c}{(c^2-b^2)E_c} \left[ -\frac{b^2 c^2 (p_0 - p')}{r} + (p' b^2 - p_0 c^2)(1-2\nu_c) r \right]$$

ASSUMPTION:

1. No end effects.

# PRESSURE - HOLLOW CYLINDER - EXTERNAL PRESSURE - NO CASE

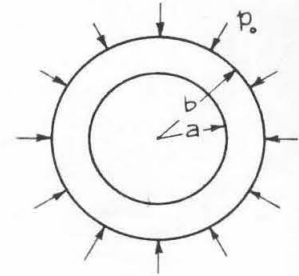
## TWO DIMENSIONAL PLANE STRESS:

BOUNDARY CONDITIONS:

$$\sigma_r = 0: \quad r = a$$

$$\sigma_r = -p_0: \quad r = b$$

$$\sigma_z = 0: \quad z = 0, l$$



STRESSES:

$$\sigma_r = -\frac{b^2 p_0}{b^2 - a^2} \left(1 - \frac{a^2}{r^2}\right)$$

$$\sigma_\theta = -\frac{b^2 p_0}{b^2 - a^2} \left(1 + \frac{a^2}{r^2}\right)$$

STRAINS & DISPLACEMENTS:

$$\left. \begin{aligned} \epsilon_r &= -\frac{b^2 p_0}{(b^2 - a^2)E} \left[ (1 - \nu) - (1 + \nu) \frac{a^2}{r^2} \right] \\ \epsilon_\theta &= -\frac{b^2 p_0}{(b^2 - a^2)E} \left[ (1 - \nu) + (1 + \nu) \frac{a^2}{r^2} \right] \end{aligned} \right\} u = -\frac{b^2 p_0}{(b^2 - a^2)E} \left[ (1 - \nu)r + (1 + \nu) \frac{a^2}{r} \right]$$

$$\epsilon_z = \frac{2\nu b^2 p_0}{(b^2 - a^2)E},$$

$$w = \frac{2\nu b^2 p_0}{(b^2 - a^2)E} z$$

## TWO DIMENSIONAL PLANE STRAIN:

Boundary conditions & stresses same except  $\epsilon_z = 0: z = 0, l$   
and  $\sigma_z = -\frac{2\nu b^2 p_0}{b^2 - a^2}$

STRAINS & DISPLACEMENTS:

$$\left. \begin{aligned} \epsilon_r &= -\frac{b^2 p_0 (1 + \nu)}{(b^2 - a^2)E} \left[ (1 - 2\nu) - \frac{a^2}{r^2} \right] \\ \epsilon_\theta &= -\frac{b^2 p_0 (1 + \nu)}{(b^2 - a^2)E} \left[ (1 - 2\nu) + \frac{a^2}{r^2} \right] \end{aligned} \right\} u = -\frac{b^2 p_0 (1 + \nu)}{(b^2 - a^2)E} \left[ (1 - 2\nu)r + \frac{a^2}{r} \right]$$

ASSUMPTION:

1. No end effects

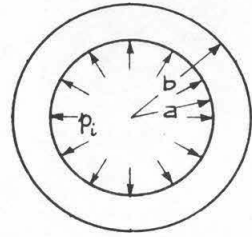
PRESSURE - HOLLOW CYLINDER - INTERNAL PRESSURE - NO CASETWO DIMENSIONAL PLANE STRESS:

BOUNDARY CONDITIONS:

$$\sigma_r = -p_i : r = a$$

$$\sigma_r = 0 : r = b$$

$$\sigma_z = 0 : z = 0, l$$



STRESSES:

$$\sigma_r = \frac{a^2 p_i}{b^2 - a^2} \left( 1 - \frac{b^2}{r^2} \right)$$

$$\sigma_\theta = \frac{a^2 p_i}{b^2 - a^2} \left( 1 + \frac{b^2}{r^2} \right)$$

STRAINS &amp; DISPLACEMENTS:

$$\left. \begin{aligned} \epsilon_r &= \frac{a^2 p_i}{(b^2 - a^2)E} \left[ (1 - \nu) - (1 + \nu) \frac{b^2}{r^2} \right] \\ \epsilon_\theta &= \frac{a^2 p_i}{(b^2 - a^2)E} \left[ (1 - \nu) + (1 + \nu) \frac{b^2}{r^2} \right] \end{aligned} \right\}$$

$$u = \frac{a^2 p_i}{(b^2 - a^2)E} \left[ (1 - \nu)r + (1 + \nu) \frac{b^2}{r} \right]$$

$$\epsilon_z = -\frac{2\nu}{E} \frac{a^2 p_i}{b^2 - a^2} ,$$

$$w = -\frac{2\nu}{E} \frac{a^2 p_i}{b^2 - a^2} z$$

TWO DIMENSIONAL STRAIN:

Boundary conditions & stresses same except  $\epsilon_z = 0 : z = 0, l$   
and  $\sigma_z = 2\nu \frac{a^2 p_i}{b^2 - a^2}$

STRAINS &amp; DISPLACEMENTS:

$$\left. \begin{aligned} \epsilon_r &= \frac{a^2 p_i (1 + \nu)}{(b^2 - a^2)E} \left[ (1 - 2\nu) - \frac{b^2}{r^2} \right] \\ \epsilon_\theta &= \frac{a^2 p_i (1 + \nu)}{(b^2 - a^2)E} \left[ (1 - 2\nu) + \frac{b^2}{r^2} \right] \end{aligned} \right\}$$

$$u = \frac{a^2 p_i (1 + \nu)}{(b^2 - a^2)E} \left[ (1 - 2\nu)r + \frac{b^2}{r} \right]$$

ASSUMPTION :

1. No end effects



PRESSURE-HOLLOW CYLINDER-EXTERNAL PRESSURE-WITH CASETWO DIMENSIONAL PLANE STRESS:

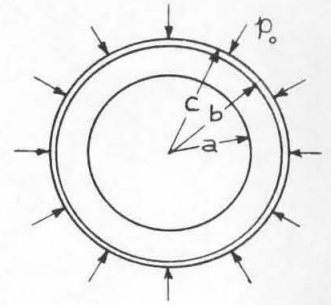
BOUNDARY CONDITIONS:

$$\sigma_r = 0: r = a$$

$$u = u_c, \tau = 0: r = b$$

$$\sigma_r = -p_o: r = c$$

$$\sigma_z = 0: z = 0, l$$



STRESSES IN HOLLOW CYLINDER:

$$\sigma_r = -\frac{b^2 p'}{b^2 - a^2} \left(1 - \frac{a^2}{r^2}\right)$$

$$p' = \frac{2c^2 p_o}{\frac{(c^2 - b^2) E_c}{(b^2 - a^2) E} \left[ (1 + \nu) a^2 + (1 - \nu) b^2 \right] + \left[ (1 + \nu_c) c^2 + (1 - \nu_c) b^2 \right]}$$

$$\sigma_\theta = -\frac{b^2 p'}{b^2 - a^2} \left(1 + \frac{a^2}{r^2}\right)$$

FOR THIN CASE:

$$p' = \frac{p_o}{1 + \left[ (1 + \nu) a^2 + (1 - \nu) b^2 \right] \frac{h E_c}{(b^2 - a^2) b E}}$$

STRAINS &amp; DISPLACEMENTS IN CYLINDER:

$$\left. \begin{aligned} \epsilon_r &= -\frac{b^2 p'}{(b^2 - a^2) E} \left[ (1 - \nu) - (1 + \nu) \frac{a^2}{r^2} \right] \\ \epsilon_\theta &= -\frac{b^2 p'}{(b^2 - a^2) E} \left[ (1 - \nu) + (1 + \nu) \frac{a^2}{r^2} \right] \end{aligned} \right\} u = \frac{-b^2 p'}{(b^2 - a^2) E} \left[ (1 - \nu) r + (1 + \nu) \frac{a^2}{r} \right]$$

$$\epsilon_z = \frac{2 \nu b^2 p'}{(b^2 - a^2) E},$$

$$w = \frac{2 \nu b^2 p'}{(b^2 - a^2) E} z$$

STRESSES IN CASE:

$$\sigma_r = \frac{b^2 c^2 (p_0 - p')}{c^2 - b^2} \frac{1}{r^2} + \frac{p' b^2 - p_0 c^2}{c^2 - b^2}$$

$$\sigma_\theta = -\frac{b^2 c^2 (p_0 - p')}{c^2 - b^2} \frac{1}{r^2} + \frac{p' b^2 - p_0 c^2}{c^2 - b^2}$$

STRAINS &amp; DISPLACEMENTS IN CASE:

$$\left. \begin{aligned} \epsilon_r &= \frac{1}{(c^2 - b^2) E_c} \left[ (1 + \nu_c) b^2 c^2 (p_0 - p') \frac{1}{r^2} + (1 - \nu_c) (p' b^2 - p_0 c^2) \right] \\ \epsilon_\theta &= \frac{1}{(c^2 - b^2) E_c} \left[ -(1 + \nu_c) b^2 c^2 (p_0 - p') \frac{1}{r^2} + (1 - \nu_c) (p' b^2 - p_0 c^2) \right] \end{aligned} \right\} u = \int \epsilon_r dr = r \epsilon_\theta$$

$$\epsilon_z = -\frac{2 \nu_c (p' b^2 - p_0 c^2)}{(c^2 - b^2) E_c},$$

$$w = -\frac{2 \nu_c (p' b^2 - p_0 c^2)}{(c^2 - b^2) E_c} z$$

ASSUMPTIONS:

1. No end effects
2. No shear between cylinder and case

PRESSURE-HOLLOW CYLINDER-EXTERNAL PRESSURE-WITH CASE-(CONT.)TWO DIMENSIONAL PLANE STRAIN:

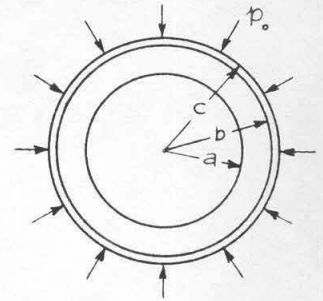
BOUNDARY CONDITIONS:

$$\sigma_r = 0: \quad r = a$$

$$u = u_c: \quad r = b$$

$$\sigma_r = -p_o: \quad r = c$$

$$\epsilon_z = 0: \quad z = 0, l$$



STRESSES IN HOLLOW CYLINDER:

$$\sigma_r = -\frac{b^2 p'}{b^2 - a^2} \left(1 - \frac{a^2}{r^2}\right),$$

$$p' = \frac{2(1-\nu_c)c^2 p_o}{\frac{(c^2 - b^2)(1+\nu)E_c}{(b^2 - a^2)(1+\nu_c)E} \left[ a^2 + (1-2\nu)b^2 \right] + \left[ c^2 + (1-2\nu_c)b^2 \right]}$$

 $p'$  = pressure between cylinder and case.

FOR THIN CASE:

$$p' = \frac{p_o}{1 + \left[ a^2 + (1-2\nu)b^2 \right] \frac{(1+\nu)hE_c}{(b^2 - a^2)(1-\nu_c^2)bE}}$$

$$\sigma_\theta = -\frac{b^2 p'}{b^2 - a^2} \left(1 + \frac{a^2}{r^2}\right)$$

$$\sigma_z = -\frac{2\nu b^2 p'}{b^2 - a^2}$$

STRAINS &amp; DISPLACEMENTS IN CYLINDER:

$$\left. \begin{aligned} \epsilon_r &= -\frac{(1+\nu)b^2 p'}{(b^2 - a^2)E} \left[ (1-2\nu) - \frac{a^2}{r^2} \right] \\ \epsilon_\theta &= -\frac{(1+\nu)b^2 p'}{(b^2 - a^2)E} \left[ (1-2\nu) + \frac{a^2}{r^2} \right] \end{aligned} \right\} u = -\frac{(1+\nu)b^2 p'}{(b^2 - a^2)E} \left[ (1-2\nu)r + \frac{a^2}{r} \right]$$

STRESSES IN CASE :

$$\sigma_r = \frac{b^2 c^2 (p_0 - p')}{c^2 - b^2} \frac{1}{r^2} + \frac{p' b^2 - p_0 c^2}{c^2 - b^2}$$

$$\sigma_\theta = -\frac{b^2 c^2 (p_0 - p')}{c^2 - b^2} \frac{1}{r^2} + \frac{p' b^2 - p_0 c^2}{c^2 - b^2}$$

$$\sigma_z = \frac{2\nu_c (p' b^2 - p_0 c^2)}{c^2 - b^2}$$

STRAINS &amp; DISPLACEMENTS IN CASE :

$$\epsilon_r = \frac{1 + \nu_c}{(c^2 - b^2) E_c} \left[ b^2 c^2 (p_0 - p') \frac{1}{r^2} + (1 - 2\nu_c)(p' b^2 - p_0 c^2) \right]$$

$$\epsilon_\theta = \frac{1 + \nu_c}{(c^2 - b^2) E_c} \left[ -b^2 c^2 (p_0 - p') \frac{1}{r^2} + (1 - 2\nu_c)(p' b^2 - p_0 c^2) \right]$$

$$u = \frac{1 + \nu_c}{(c^2 - b^2) E_c} \left[ -b^2 c^2 (p_0 - p) \frac{1}{r} + (1 - 2\nu_c)(p' b^2 - p_0 c^2) r \right]$$

ASSUMPTION :

1. No end effects.

PRESSURE-HOLLOW CYLINDER-INTERNAL PRESSURE-WITH CASETWO DIMENSIONAL PLANE STRESS:

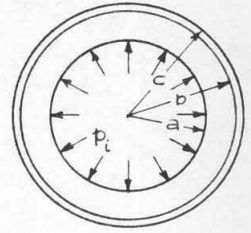
BOUNDARY CONDITIONS:

$$\sigma_r = -p_i : r = a$$

$$u = u_c, \tau = 0 : r = b$$

$$\sigma_r = 0 : r = c$$

$$\sigma_z = 0 : z = 0, l$$



STRESSES IN HOLLOW CYLINDER:

$$\sigma_r = \frac{a^2 b^2 (p' - p_i)}{b^2 - a^2} \frac{1}{r^2} + \frac{p_i a^2 - p' b^2}{b^2 - a^2}$$

$$\sigma_\theta = -\frac{a^2 b^2 (p' - p_i)}{b^2 - a^2} \frac{1}{r^2} + \frac{p_i a^2 - p' b^2}{b^2 - a^2}$$

$$p' = \frac{2 a^2 p_i}{\left[ (1+\nu) a^2 + (1-\nu) b^2 \right] + \frac{(b^2 - a^2) E}{(c^2 - b^2) E_c} \left[ (1+\nu_c) c^2 + (1-\nu_c) b^2 \right]}$$

FOR THIN CASE:

$$p' = \frac{2 a^2 p_i}{\left[ (1+\nu) a^2 + (1-\nu) b^2 \right] + \frac{(b^2 - a^2) b E}{h E_c}}$$

STRAINS &amp; DISPLACEMENTS IN CYLINDER:

$$\left. \begin{aligned} \epsilon_r &= \frac{1}{(b^2 - a^2) E} \left[ (1+\nu) a^2 b^2 (p' - p_i) \frac{1}{r^2} + (1-\nu) (p_i a^2 - p' b^2) \right] \\ \epsilon_\theta &= \frac{1}{(b^2 - a^2) E} \left[ -(1+\nu) a^2 b^2 (p' - p_i) \frac{1}{r^2} + (1-\nu) (p_i a^2 - p' b^2) \right] \end{aligned} \right\} u = \int \epsilon_r dr = r \epsilon_\theta$$

$$\epsilon_z = -\frac{2\nu (p_i a^2 - p' b^2)}{(b^2 - a^2) E},$$

$$w = -\frac{2\nu (p_i a^2 - p' b^2)}{(b^2 - a^2) E} z$$



STRESSES IN CASE:

$$\sigma_r = \frac{b^2 p'}{c^2 - b^2} \left( 1 - \frac{c^2}{r^2} \right)$$

$$\sigma_\theta = \frac{b^2 p'}{c^2 - b^2} \left( 1 + \frac{c^2}{r^2} \right)$$

STRAINS &amp; DISPLACEMENTS IN CASE:

$$\left. \begin{aligned} \epsilon_r &= \frac{b^2 p'}{(c^2 - b^2) E_c} \left[ (1 - \nu_c) - (1 + \nu_c) \frac{c^2}{r^2} \right] \\ \epsilon_\theta &= \frac{b^2 p'}{(c^2 - b^2) E_c} \left[ (1 - \nu_c) + (1 + \nu_c) \frac{c^2}{r^2} \right] \end{aligned} \right\} u = \frac{b^2 p'}{(c^2 - b^2) E_c} \left[ (1 - \nu_c) r + (1 + \nu_c) \frac{c^2}{r} \right]$$

$$\epsilon_z = - \frac{2 \nu_c b^2 p'}{(c^2 - b^2) E_c}$$

$$w = - \frac{2 \nu_c b^2 p'}{(c^2 - b^2) E_c} z$$

ASSUMPTIONS:

1. No end effects.
2. No shear between cylinder and case.

## PRESSURE-HOLLOW CYLINDER-INTERNAL PRESSURE-WITH CASE-(CONT.)

TWO DIMENSIONAL PLANE STRAIN:

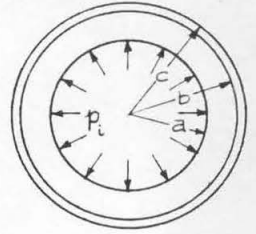
BOUNDARY CONDITIONS:

$$\sigma_r = -p_i : r = a$$

$$u = u_c : r = b$$

$$\sigma_r = 0 : r = c$$

$$\epsilon_z = 0 : z = 0, l$$



STRESSES IN HOLLOW CYLINDER:

$$\sigma_r = \frac{a^2 b^2 (p' - p_i)}{b^2 - a^2} \frac{1}{r^2} + \frac{p_i a^2 - p' b^2}{b^2 - a^2}$$

$$\sigma_\theta = - \frac{a^2 b^2 (p' - p_i)}{b^2 - a^2} \frac{1}{r^2} + \frac{p_i a^2 - p' b^2}{b^2 - a^2}$$

$$\sigma_z = \frac{2\nu(p_i a^2 - p' b^2)}{b^2 - a^2}$$

$$p' = \frac{2(1-\nu) a^2 p_i}{\left[ a^2 + (1-2\nu) b^2 \right] + \frac{(b^2 - a^2)(1+\nu_c) E}{(c^2 - b^2)(1+\nu) E_c} \left[ c^2 + (1-2\nu_c) b^2 \right]}$$

 $p'$  = pressure between cylinder and case.

FOR THIN CASE:

$$p' = \frac{2(1-\nu) a^2 p_i}{\left[ a^2 + (1-2\nu) b^2 \right] + \frac{(b^2 - a^2)(1-\nu_c) b E}{(1+\nu) h E_c}}$$

STRAINS &amp; DISPLACEMENTS IN CYLINDER:

$$\left. \begin{aligned} \epsilon_r &= \frac{1+\nu}{(b^2 - a^2) E} \left[ a^2 b^2 (p' - p_i) \frac{1}{r^2} + (1-2\nu)(p_i a^2 - p' b^2) \right] \\ \epsilon_\theta &= \frac{1+\nu}{(b^2 - a^2) E} \left[ -a^2 b^2 (p' - p_i) \frac{1}{r^2} + (1-2\nu)(p_i a^2 - p' b^2) \right] \end{aligned} \right\} u = \int \epsilon_r dr = r \epsilon_\theta$$

STRESSES IN CASE:

$$\sigma_r = \frac{b^2 p'}{c^2 - b^2} \left(1 - \frac{c^2}{r^2}\right)$$

$$\sigma_\theta = \frac{b^2 p'}{c^2 - b^2} \left(1 + \frac{c^2}{r^2}\right)$$

$$\sigma_z = \frac{2 \nu_c b^2 p'}{c^2 - b^2}$$

STRAINS &amp; DISPLACEMENTS IN CASE:

$$\left. \begin{aligned} \epsilon_r &= \frac{(1 + \nu_c) b^2 p'}{(c^2 - b^2) E_c} \left[ (1 - 2 \nu_c) - \frac{c^2}{r^2} \right] \\ \epsilon_\theta &= \frac{(1 + \nu_c) b^2 p'}{(c^2 - b^2) E_c} \left[ (1 - 2 \nu_c) + \frac{c^2}{r^2} \right] \end{aligned} \right\} u = \frac{(1 + \nu_c) b^2 p'}{(c^2 - b^2) E_c} \left[ (1 - 2 \nu_c) r + \frac{c^2}{r} \right]$$

ASSUMPTION:

1. No end effects

# TORSION EQUATIONS - ST. VENANT

## ASSUMPTIONS:

1. No dependence on  $\theta$  coordinate (not same as twist:  $\theta = \frac{\alpha}{b}$ )
2.  $\sigma_r = \sigma_z = \tau_{r\theta} = \tau_{rz} = 0$ , no body forces
3.  $\epsilon_z = 0$  (or  $\theta = \frac{\alpha}{b} \ll 1$ )

## EQUILIBRIUM:

$$\frac{\partial \tau_{\theta z}}{\partial z} = 0$$

## STRESS - STRAIN:

$$\gamma_{\theta z} = \frac{\tau_{\theta z}}{\mu}$$

## STRAIN - DISPLACEMENT:

$$\gamma_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r} = 0,$$

$$\gamma_{\theta z} = \frac{\partial v}{\partial z}$$

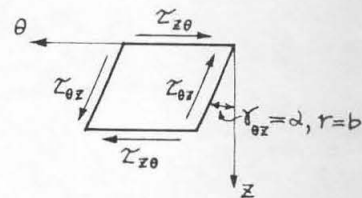
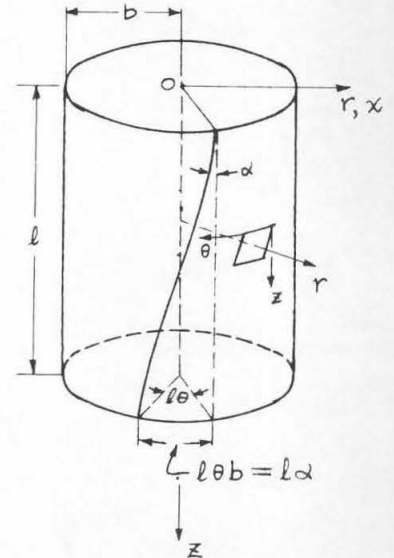
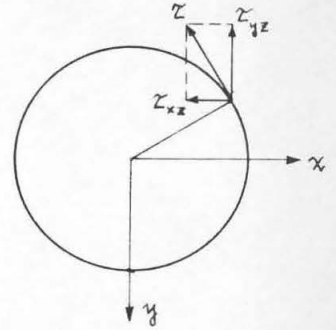
## BOUNDARY CONDITIONS:

1. Ends ( $z=0, l$ ) rotate as solid disks  $\Rightarrow v$  linear in  $r$ .
2. Cylindrical surface ( $r=b$ ) uniformly twisted  $\Rightarrow \tau_{\theta z}$  const., for  $r=b$ .

## CARTESIAN COORDINATE FORM:

$$\tau_{yz} = \mu \theta x, \quad \tau_{xz} = -\mu \theta y, \quad \sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$$

$$u = -\theta z y, \quad v = \theta z x, \quad w = 0$$



TORSION EQUATIONS - SECOND ORDER

## ASSUMPTIONS:

1. No dependence on  $\theta$  coordinate
2.  $\sigma_r = \sigma_z = \tau_{r\theta} = \tau_{rz} = 0$
3. No body forces

FROM RIVLIN'S NOTES,\* PAGE 81 & 83:

$$\epsilon_z = \frac{\theta^2}{4Aa_1(a_1 + 3a_2)} \left\{ B \left[ a_5(a_1 + 2a_2) - a_1(2a_2 - a_3) \right] \right\}$$

where the  $a_i$  are coefficients of the energy expression:

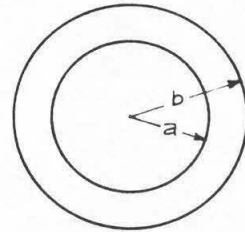
$$W = a_1 J_2 + a_2 J_1^2 + a_3 J_1 J_2 + a_4 J_1^3 + a_5 J_3$$

where the  $J_i$  are invariants of increasingly higher orders.

## CIRCULAR HOLLOW CYLINDER:

$$A = \pi (b^2 - a^2)$$

$$B = I = \pi \frac{(b^4 - a^4)}{2}$$



## ELASTIC SOLUTIONS:

$$a_1 = -\frac{E}{4(1+\nu)}$$

$$a_2 = \frac{E}{8} \frac{1-\nu}{(1+2\nu)(1-2\nu)}$$

$$a_3 = a_4 = a_5 = 0$$

$$\epsilon_z = -\frac{\theta^2}{4} \frac{(1-\nu)}{(1+\nu)} \left( \frac{b^4 - a^4}{b^2 - a^2} \right)$$

\* RIVLIN,



TORSION - SOLID OR HOLLOW CYLINDER - NO CASE

BOUNDARY CONDITIONS:

$$\tau = \tau_{z\theta} = \mu \theta r = \frac{\mu \alpha r}{b},$$

$$\sigma_z = \tau_{zr} = 0: z=0, l$$

$$\sigma_r = \tau_{rz} = \tau_{r\theta} = 0: r=b (\Rightarrow \sigma_\theta = 0)$$

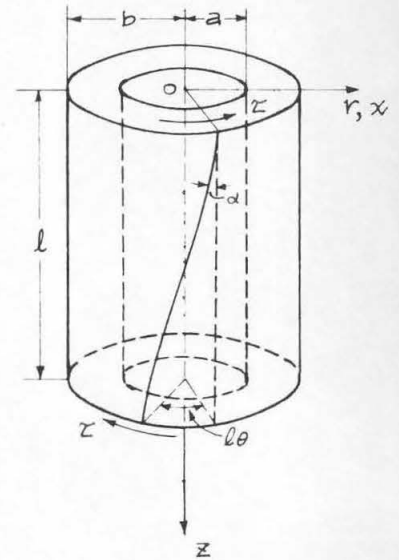
STRAINS &amp; DISPLACEMENTS:

$$\gamma = \gamma_{z\theta} = \frac{\tau}{\mu} = \theta r$$

$$\epsilon_r = \epsilon_\theta = \epsilon_z = \gamma_{rz} = \gamma_{r\theta} = 0$$

$$v = \int \gamma_{z\theta} dz = \theta r z$$

$$u = w = 0$$



TORQUE:

$$M_t = \frac{\pi \mu \theta}{2} (b^4 - a^4)$$

ASSUMPTIONS:

1. End surfaces ( $z=0, l$ ) rotate as solid disks
2. Surfaces  $z = \text{const.}$  remain plane
3.  $\epsilon_z = 0 \Rightarrow \theta \ll 1$

SECOND ORDER DISPLACEMENT:

$$\epsilon_z = - \left( \frac{1-\nu}{1+\nu} \right) \frac{b^2}{4} \theta^2$$

# TORSION - SOLID OR HOLLOW CYLINDER - WITH CASE CYLINDER AND CASE UNDER TORSION

BOUNDARY CONDITIONS:

$$\tau = \tau_{z\theta} = \bar{\mu}\theta r, \quad \sigma_z = \tau_{zr} = 0 : z=0, l$$

$$\sigma_r = \tau_{rz} = \tau_{r\theta} = 0 : r=c (\Rightarrow \sigma_\theta = 0)$$

STRAINS & DISPLACEMENTS:

$$\gamma = \theta r, \quad v = \theta r z$$

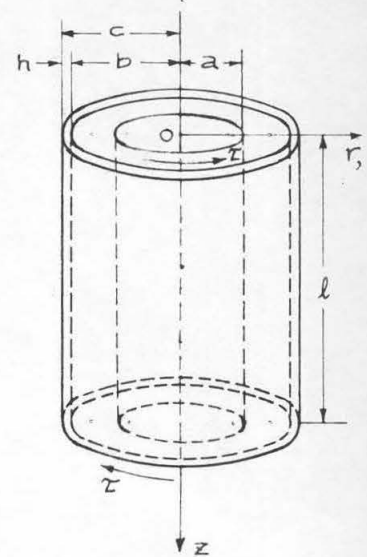
$$\epsilon_r = \epsilon_\theta = \epsilon_z = \gamma_{rz} = \gamma_{r\theta} = u = w = 0$$

TORQUE:

$$M = M_t + M_c$$

$$= \frac{\pi \mu \theta}{2} (b^4 - a^4) + \frac{\pi \mu_c \theta}{2} (c^4 - b^4)$$

$$\approx \frac{\pi \mu \theta}{2} (b^4 - a^4) + 2\pi \mu_c \theta h b^3, \quad \text{for thin case, } (h \equiv c - b)$$



EQUIVALENT SHEAR MODULUS:

$$\bar{\mu} \equiv \frac{b^4 - a^4}{c^4 - a^4} \mu + \frac{c^4 - b^4}{c^4 - a^4} \mu_c$$

$$\approx \mu + \frac{4b^3 h}{b^4 - a^4} \mu_c, \quad \text{for thin case}$$

$$\approx \mu + \frac{4h}{b} \mu_c, \quad \text{for solid cylinder & thin case}$$

ASSUMPTIONS:

1. End surfaces ( $z=0, l$ ) rotate as solid disks.
2. Surfaces  $z=\text{const.}$  remain plane.
3.  $\epsilon_z = 0 \Rightarrow \theta \ll 1$
4. No normal ( $\sigma_r$ ) interaction between cylinder and case.

# TORSION - SOLID OR HOLLOW CYLINDER - WITH CASE CASE UNDER TORSION - CYLINDER UNBONDED

BOUNDARY CONDITIONS:

$$\tau = \tau_{z\theta} = \mu_c \theta r, \quad \sigma_z = \tau_{zr} = 0:$$

$$z = 0, l \quad \& \quad b \leq r \leq c \text{ (case only)}$$

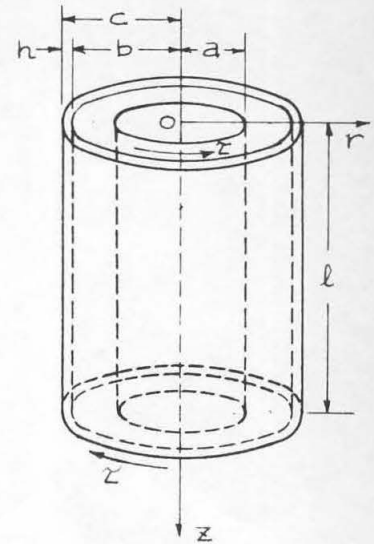
$$\sigma_r = \tau_{rz} = \tau_{r\theta} = 0: \quad r = c$$

All stresses zero on cylinder in linear theory.

STRAINS & DISPLACEMENTS OF CASE:

$$\gamma_{z\theta} = \theta r, \quad v = \theta r z,$$

$$\epsilon_r = \epsilon_\theta = \epsilon_z = \gamma_{rz} = \gamma_{r\theta} = u = w = 0$$



TORQUE:

$$M = M_c = \frac{\pi \mu_c \theta}{2} (c^4 - b^4)$$

$$\approx 2\pi \mu_c \theta h b^3, \text{ for thin case}$$

ASSUMPTIONS:

1. Torque applied only to case.
2. Cylinder in no way receives shear from case or end plates.
3.  $\epsilon_z = 0$ , hence  $\theta \ll 1$

SECOND ORDER THEORY:

$$\epsilon_z \approx -\left(\frac{1-\nu_c}{1+\nu_c}\right) \frac{b^2}{2} \theta^2, \quad \text{and if endplates are flat they impose a stress on the cylinder:}$$

$$\sigma_z = \epsilon_z E, \quad \text{assuming } E \ll E_c \text{ and no shear transmitted.}$$

# TORSION - SOLID OR HOLLOW CYLINDER-WITH CASE CASE UNDER TORSION - CYLINDER BONDED TO CASE

BOUNDARY CONDITIONS:

$$\tau_{z\theta} = \bar{\mu} \theta r, \quad \sigma_z = \tau_{zr} = 0:$$

$$z = 0, l \quad \& \quad b \leq r \leq c \quad (\text{only to case})$$

$$\sigma_r = \tau_{rz} = \tau_{r\theta} = 0: \quad r = c$$

$$v_c = v = \theta b z: \quad r = b$$

NOTE: This is identical to torsion applied to both case and cylinder at  $z = 0, l$ .

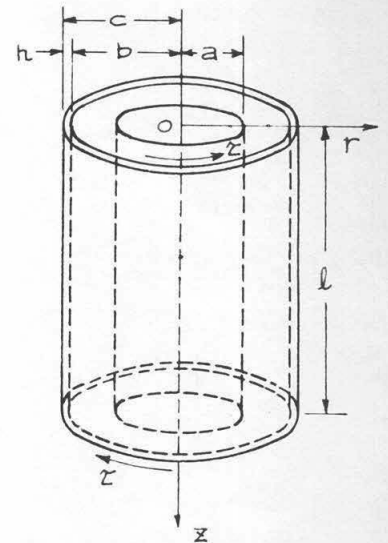
STRAIN & DISPLACEMENT:

$$\gamma_{\theta z} = \theta r,$$

$$v = \theta r z$$

TORQUE:

$$M \approx \frac{\pi \mu \theta}{2} (b^4 - a^4) + 2\pi \mu_c \theta h b^3, \quad \text{thin case}$$



EQUIVALENT SHEAR MODULUS:

$$\bar{\mu} \approx \mu + \frac{4b^3 h}{b^4 - a^4} \mu_c, \quad \text{thin case}$$

$$\approx \mu + \frac{4h}{b} \mu_c \quad \text{for solid cylinder \& thin case}$$

# TORSION - SOLID OR HOLLOW CYLINDER - WITH CASE CYLINDER UNDER TORSION - UNBONDED

BOUNDARY CONDITIONS:

$$\tau_{z\theta} = \mu \theta r, \quad \sigma_z = \tau_{zr} = 0:$$

$$z = 0, l \quad \& \quad 0 \leq r \leq b \text{ (cylinder only)}$$

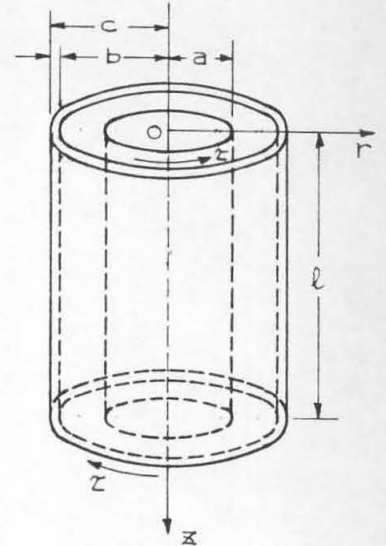
$$\sigma_r = \tau_{rz} = \tau_{r\theta} = 0: \quad r = c$$

All stresses zero on case in linear theory.

STRAINS & DISPLACEMENTS OF cylinder:

$$\gamma_{z\theta} = \theta r, \quad v = \theta r z,$$

$$\epsilon_r = \epsilon_\theta = \epsilon_z = \gamma_{rz} = \gamma_{r\theta} = u = w = 0$$



TORQUE:

$$M = \frac{\pi \mu \theta}{2} (b^4 - a^4)$$

ASSUMPTIONS:

1. Torque applied only to cylinder.
2. Case in no way receives shear from cylinder or endplates.
3.  $\epsilon_z = 0$ , hence  $\theta \ll 1$

SECOND ORDER THEORY:

$$\epsilon_z = - \left( \frac{1-\nu}{1+\nu} \right) \left( \frac{b^4 - a^4}{b^2 - a^2} \right) \frac{\theta^2}{4}$$

NOTE: If cylinder is nonelastic, this strain is different. In fact, Rivlin has found this to be positive for rubber.